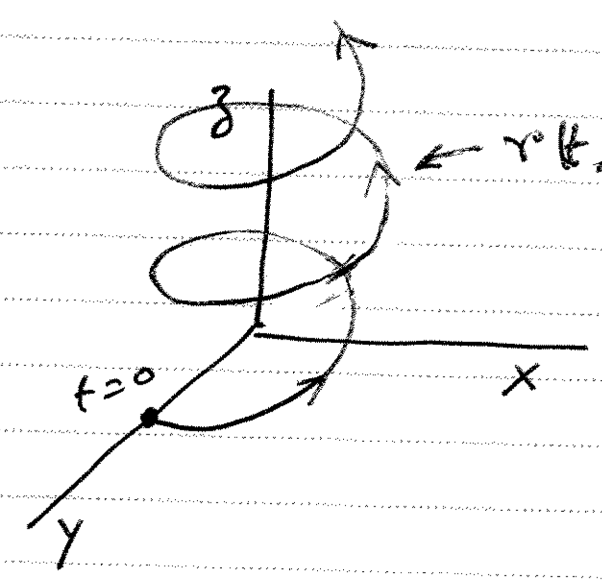
Why study vectors

A. In Physics to talk about position, velocity and acceleration of a point object.

Ex: 1

$$r(t) = \begin{pmatrix} \sin t \\ \cos t \\ t \end{pmatrix}$$ be the position

of a particle at time instant t.



$v(t) = \dot{r}(t)$ ← Velocity

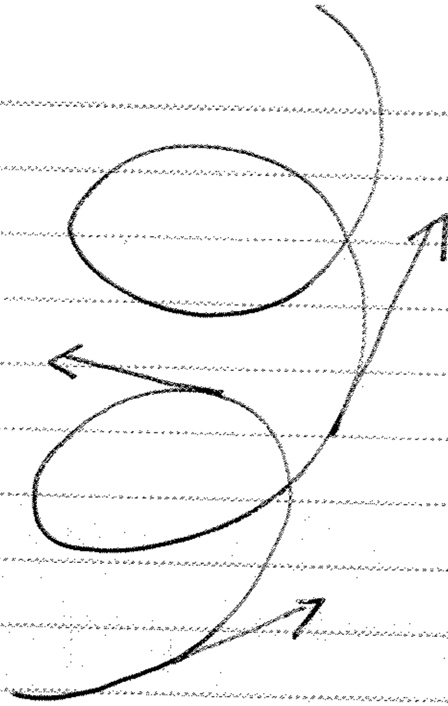
$a(t) = \ddot{r}(t)$ ← acceleration.

$$v(t) = \begin{pmatrix} \cos t \\ -\sin t \\ 1 \end{pmatrix}$$

speed $\dot{s}(t) = \|v(t)\| = \sqrt{2}$

$$a(t) = \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix}$$

The velocity vector is always pointed tangential to the curve shown in Fig.



velocity vector is shown tangent to the curve.

Define $u(t) = \dfrac{v(t)}{\|v(t)\|}$

$u(t)$ is the unit velocity vector or the unit tangent vector.

$$\|v(t)\| = \sqrt{\cos^2 t + \sin^2 t + 1}$$

$$= \sqrt{2}$$

$$u(t) = \begin{pmatrix} \cos t/\sqrt{2} \\ -\sin t/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

The acc^{n} vector $a(t)$ is perpendicular to $u(t)$. It is called the normal acc^{n}

$$a_N(t) = \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix}$$

The tangential component of the acc^{n} vector is zero.

Remark: This is not always the case but only for this problem.

Ex 2

$$r(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$$ ← position vector

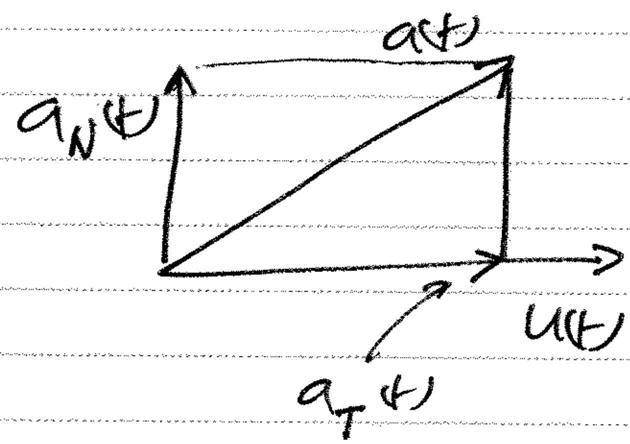
$$v(t) = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}$$ ← velocity vector.

$$\|v(t)\| = \sqrt{1+4t^2+9t^4}$$

$$u(t) = \begin{pmatrix} (1+4t^2+9t^4)^{-1/2} \\ 2t\,(1+4t^2+9t^4)^{-1/2} \\ 3t^2(1+4t^2+9t^4)^{-1/2} \end{pmatrix}$$

unit tangent vector.

In general, the acc$^{\underline{n}}$ vector does not point in the tangential direction.



Define

$$a_T(t) = \text{proj}_{[u(t)]}\, a(t)$$ ← tangential component of the acc$^{\underline{n}}$.

$$a_N(t) = a(t) - a_T(t)$$ ← Normal component of the acc$^{\underline{n}}$.

$$a(t) = \begin{pmatrix} 0 \\ 2 \\ 6t \end{pmatrix}$$

Define

$$a_T(t) = \text{proj}_{[v(t)]} a(t)$$

$$a_T(t) = \alpha \, v(t)$$

$$[a - \alpha \, v(t)] \cdot [v(t)] = 0$$

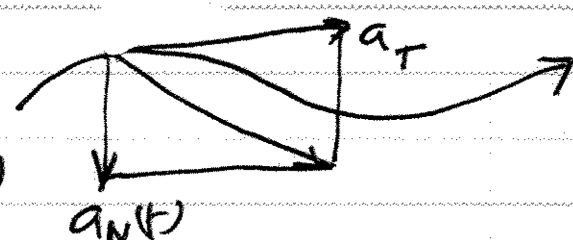
$$\alpha = \frac{a \cdot v}{v \cdot v} \qquad a_T(t) = \frac{a \cdot v}{v \cdot v} v$$

$$a \cdot v = 4t + 18t^3$$

$$v \cdot v = 1 + 4t^2 + 9t^4$$

$$a_T(t) = \frac{2t(2 + 9t^2)}{1 + 4t^2 + 9t^4} \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}$$

a_N



$$a_N(t) = a - a_T(t)$$

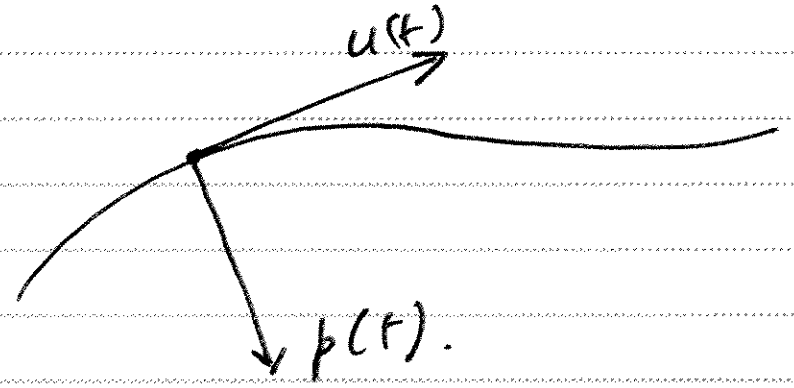
$$= \begin{pmatrix} -\dfrac{2t(2+9t^2)}{1+4t^2+9t^4} \\ 2-\dfrac{4t^2(2+9t^2)}{1+4t^2+9t^4} \\ 6t-\dfrac{6t^3(2+9t^2)}{1+4t^2+9t^4} \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{2t(2+9t^2)}{1+4t^2+9t^4} \\ \dfrac{2(1-9t^4)}{1+4t^2+9t^4} \\ \dfrac{6t(1+2t^2)}{1+4t^2+9t^4} \end{pmatrix}$$

unit normal vector

$$p(t) = \frac{a_N(t)}{\|a_N(t)\|}$$

$$= \frac{\begin{pmatrix} -t(2+9t^2) \\ 1-9t^4 \\ 3t(1+2t^2) \end{pmatrix}}{\sqrt{1+13t^2+54t^4+117t^6+81t^8}}$$



Example 1 (continued)

Note that

$$u(t) = \begin{pmatrix} \cos t/\sqrt{2} \\ -\sin t/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \leftarrow \text{unit tangent vector}$$

$$p(t) = \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix} \leftarrow \text{unit normal vector.}$$

If we compute $\dot{u}$ we obtain

$$\dot{u}(t) = \frac{1}{\sqrt{2}}\, p(t)$$

Thus $\dot{u}(t)$ is in the same direction as $p(t)$. $\frac{1}{\cancel{\sqrt{2}}}$ is the curvature $= 1/2$

$\sqrt{2}$

In general we write

$$\dot{u}(t) = \kappa(t)\, p(t)$$

curvature = $\kappa / \dot{s}$

~~curvature~~.

$\dot{s} = \sqrt{2}$

In this case we have

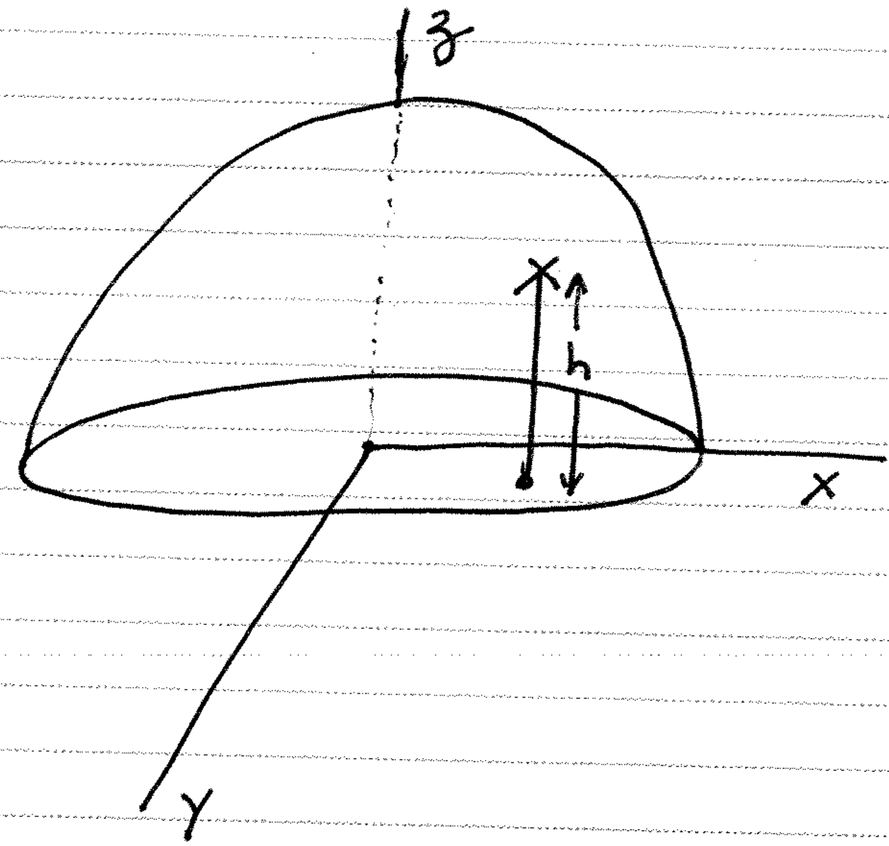
$$\kappa(t) = 1/\sqrt{2}$$

curvature $= 1/2$

Remark: A straight line has zero curvature. A circle has constant curvature.

"curvature" together with another quantity called "torsion" measures the shape of a curve in $\mathbb{R}^3$.

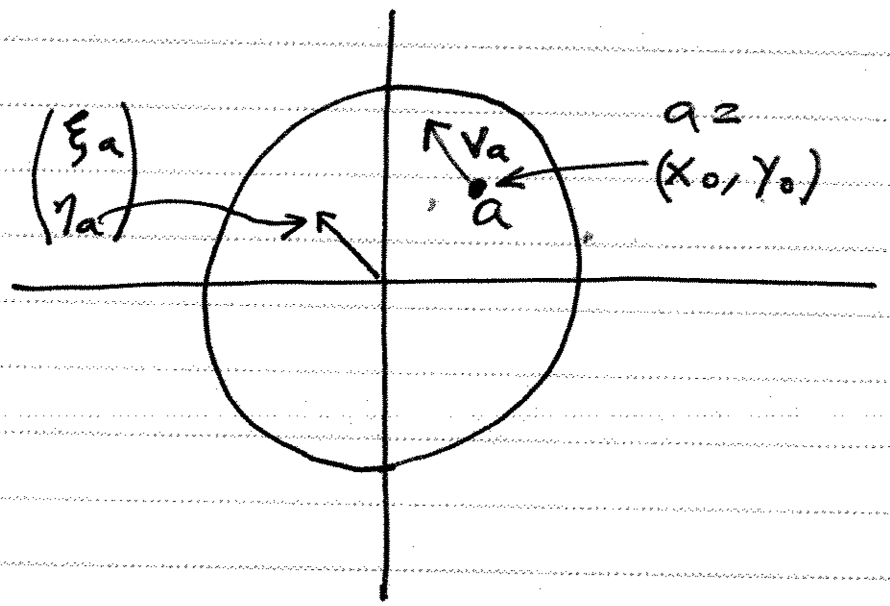
B Vectors are useful in calculating directional derivatives.



Consider the top hemisphere of a sphere of radius R centered at (0,0,0). The height of the sphere at any point (x, y) is given by

$$h(x,y) = \sqrt{R^2 - x^2 - y^2}$$

(See figure)



We are interested in the rate at which the height changes along any direction v_a starting at

$$a = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

In co-ordinates let us write

$$V_a = \begin{pmatrix} \xi_a \\ \eta_a \end{pmatrix} \quad \text{(See Figure)}$$

Note that in order to write this we need to translate V_a to the origin.

Def: The directional derivative of the function $h(x,y)$ in the direction V_a at the point a is given by

$$V_a h(x_0, y_0) = \xi_a \frac{\partial h(x_0, y_0)}{\partial x} + \eta_a \frac{\partial h}{\partial y}(x_0, y_0)$$

Note that

$$\left.\frac{\partial h}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}} = -\frac{x_0}{\sqrt{R^2 - x_0^2 - y_0^2}}$$

$$\left.\frac{\partial h}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}} = -\frac{y_0}{\sqrt{R^2 - x_0^2 - y_0^2}}$$

$$\therefore\ V_a h(x_0, y_0) = -\frac{\xi_a x_0 + \eta_a y_0}{\sqrt{R^2 - x_0^2 - y_0^2}}$$

The directional derivative measures the rate at which a function $h(x,y)$ changes at a pt. 'a' in a given direction V_a.

Many often, one choose V_a to be an unit vector

$$V_a = \begin{pmatrix} \xi_a \\ \eta_a \end{pmatrix} \qquad \xi_a^2 + \eta_a^2 = 1$$

describing only direction.

The vector

$$\begin{pmatrix} \frac{\partial h}{\partial x}(x_0, y_0) \\ \\ \frac{\partial h}{\partial y}(x_0, y_0) \end{pmatrix}$$

is called the gradient of h at the point (x_0, y_0) and is denoted by

$\nabla h(x_0, y_0)$.

Using this notation we can write

$$V_a h(x_0, y_0) = \nabla h(x_0, y_0) \cdot V_a$$

which is the dot product of the gradient vector and the direction vector.

Directional derivative can be positive or negative

+ve indicates that the function is increasing along the direction V_a

−ve indicates that the function is decreasing along the direction V_a.

Q: Assume V_a is an unit vector. Find V_a such that $V_a h(x_0, y_0)$ is maximum.

A: $$V_a h(x_0,y_0) = \|\nabla h(x_0,y_0)\| \underbrace{\|V_a\|}_{=1} \cos\theta$$

$$= \|\nabla h(x_0,y_0)\| \cos\theta$$

For $V_a h(x_0,y_0)$ to be maximum we

have $\theta = 0$, $\cos\theta = 1$

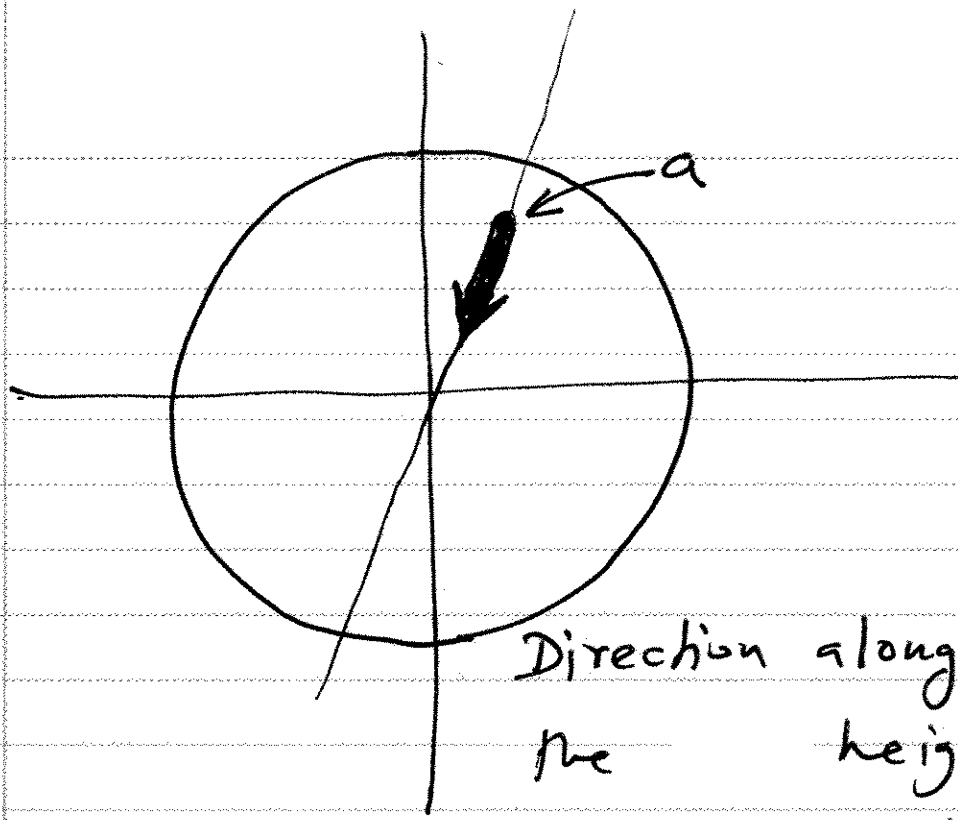
Hence

$$V_a = \frac{\nabla h(x_0, y_0)}{\| \nabla h(x_0, y_0) \|}$$

For our problem

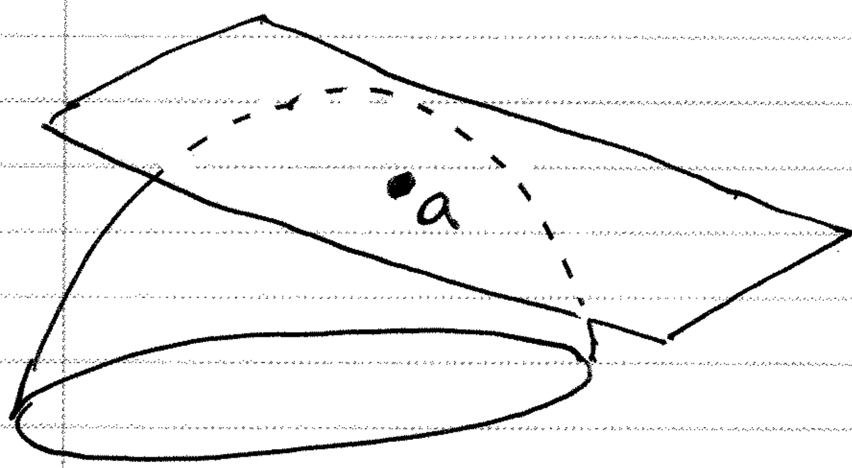
$$\nabla h(x_0, y_0) = -\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \Big/ \sqrt{R^2 - x_0^2 - y_0^2}$$

$$\frac{\nabla h(x_0, y_0)}{\| h(x_0, y_0) \|} = -\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \Big/ \sqrt{x_0^2 + y_0^2}$$



Direction along which the height increases most rapidly.

C. Vectors are useful is describing tangent planes to a surface



Let $a = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ be a pt. on the top hemisphere of a sphere

$$x^2 + y^2 + z^2 = R^2$$

Define $\phi(x, y, z) = x^2 + y^2 + z^2 - R^2$

The sphere is given by

$$\phi(x, y, z) = 0.$$

Our goal is to approximate the sphere upto a tangent plane at point a. What is the equation of the tangent plane.

If we write

$$\phi(x,y,z) = \overbrace{\phi(x_0, y_0, z_0)}^{=0} + \left.\frac{\partial \phi}{\partial x}\right|_a (x - x_0) + \left.\frac{\partial \phi}{\partial y}\right|_a (y - y_0) + \left.\frac{\partial \phi}{\partial z}\right|_a (z - z_0) + h.o.t.$$

it follows that up to terms linear in x, y & z we write

$$\phi(x,y,z) \approx \nabla \phi(x_0,y_0,z_0) \cdot \begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix}$$

where

$\nabla \phi$ is the famous gradient vector

$$\nabla \phi = \begin{pmatrix} \partial \phi/\partial x \\ \partial \phi/\partial y \\ \partial \phi/\partial z \end{pmatrix}$$

For our problem.

$$\nabla \phi(x_0, y_0, z_0) = \begin{pmatrix} 2x_0 \\ 2y_0 \\ 2z_0 \end{pmatrix}$$

Hence

$$\phi(x,y,z) \simeq 2x_0(x-x_0) + 2y_0(y-y_0) + 2z_0(z-z_0).$$

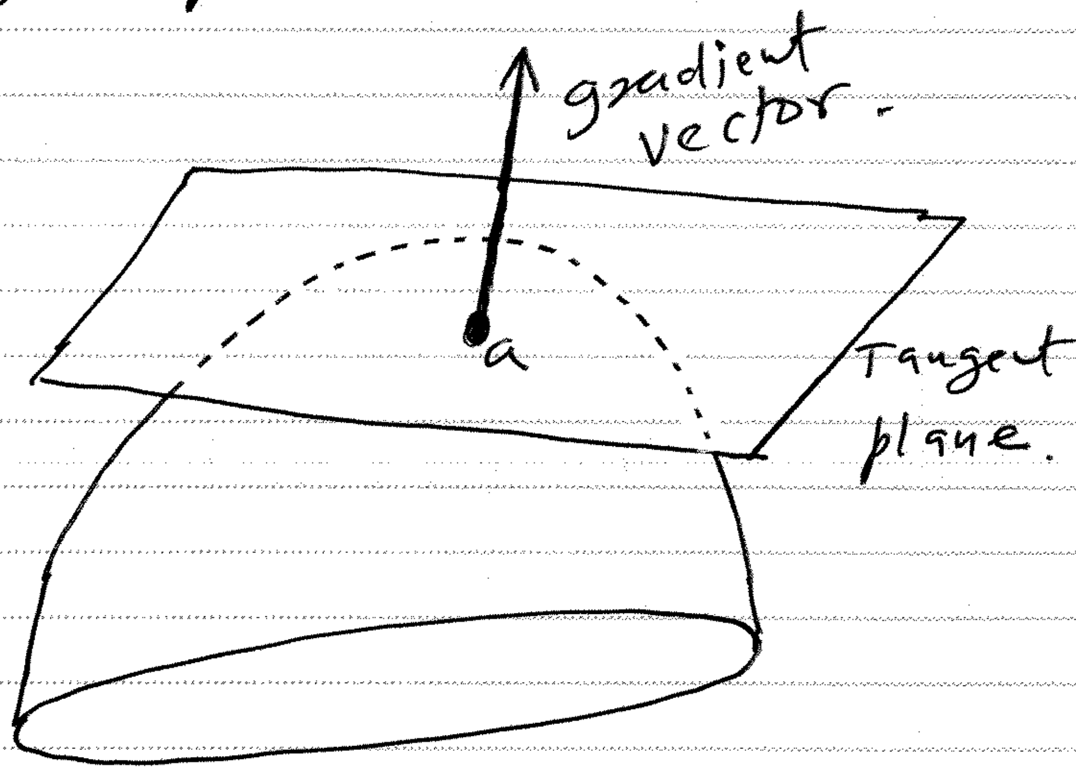
Tangent plane is given by

$$2x_0(x-x_0) + 2y_0(y-y_0) + 2z_0(z-z_0) = 0$$

$$\Rightarrow x_0 x + y_0 y + z_0 z = x_0^2 + y_0^2 + z_0^2$$

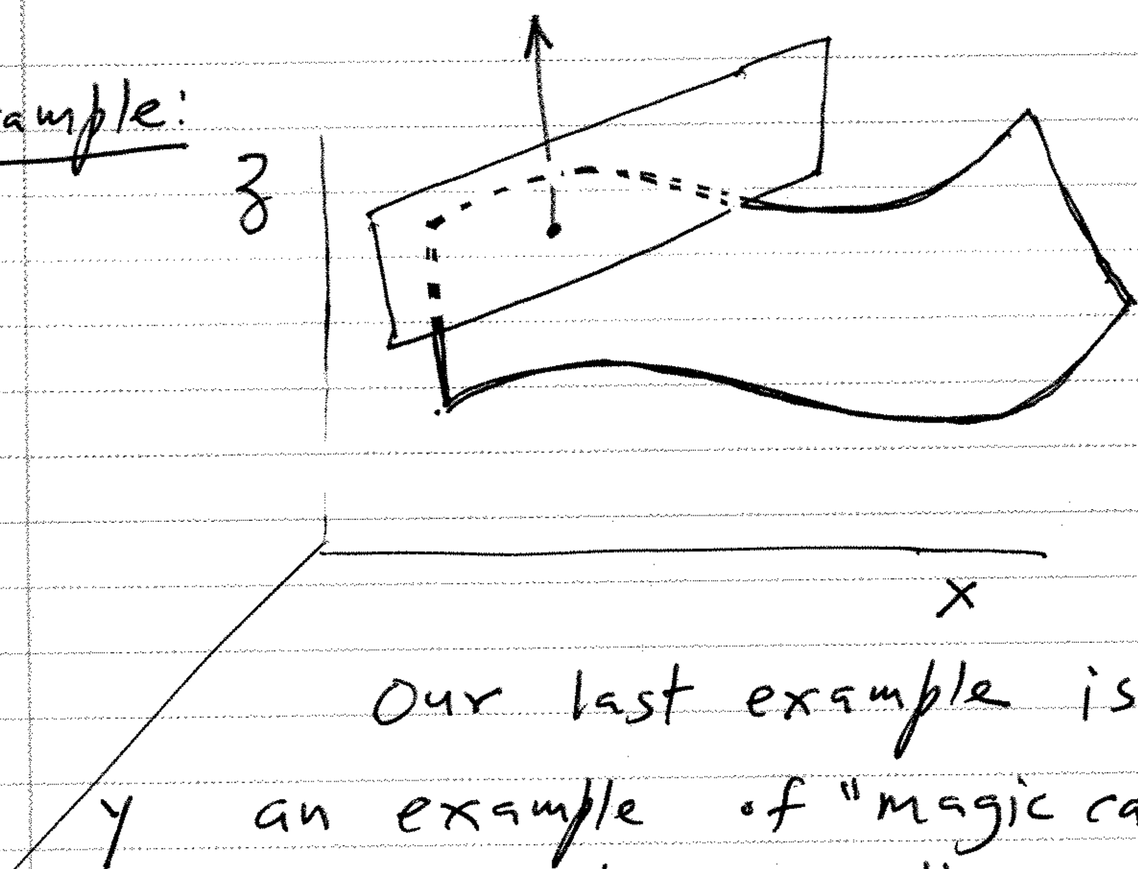
Eqn of the tangent plane $= R^2$

Note that the gradient vector is a vector pointing outwards and is perpendicular to the tangent plane.



<u>Gradient vector is an outward normal vector.</u>

Example:



Our last example is an example of "magic carpet" given by the equation

$$z = S(x, y).$$

Here

$$\phi(x,y,z) = z - S(x,y)$$

$$\nabla \phi = \begin{pmatrix} \partial \phi/\partial x \\ \partial \phi/\partial y \\ \partial \phi/\partial z \end{pmatrix} = \begin{pmatrix} -\partial S/\partial x \\ -\partial S/\partial y. \\ 1 \end{pmatrix}$$

$$a =$$

At any pt. $(x_0, y_0, S(x_0, y_0))$ on the carpet,

$$\nabla \phi(a) = \begin{pmatrix} -\frac{\partial S}{\partial x}(x_0, y_0) \\ -\frac{\partial S}{\partial y}(x_0, y_0) \\ 1 \end{pmatrix}$$

Tangent plane is given by

$$-\frac{\partial S}{\partial x}(x_0, y_0)(x - x_0) - \frac{\partial S}{\partial y}(x_0, y_0)(y - y_0) + 1(z - z_0) = 0$$

$$\Rightarrow \boxed{(z - z_0) = \frac{\partial S}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial S}{\partial x}(x_0, y_0)(y - y_0)}$$

Tangent plane to the carpet.